Prime Decomposition Chains

The 'Sum Of Prime Factors' Function Iterates To Form Sequences Of Sums

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Prime Decompositions and *sopf*(*n*)

The *prime decomposition* of a number entails its factorization into its constituent primes (also called prime *factorization*). A formal (Mathworld) definition says:

> Given a positive integer $n \geq 2$, the prime factorization is written
>
> $$n = p_1^{a_1} \cdot p_2^{a_2} \ldots \cdot p_k^{a_k}$$
>
> where the p_is are the prime factors, each of order a_i. Each factor $p_i^{a_i}$ is called a primary.

The *sum of prime factors* function *sopf*(*n*) = *g*(*f*(*n*)), where $f(n) = p_1^{a_1} \cdot p_2^{a_2} \ldots \cdot p_k^{a_k}$ is followed by $g(p_1^{a_1} \cdot p_2^{a_2} \ldots \cdot p_k^{a_k}) = a_1 p_1 + a_2 p_2 \ldots + a_k p_k = n'$. That is, *g* "downgrades" the operators in the decomposition. E.g., *sopf*(18) = 2 + 3 · 2 = 8. (See Sloane's A008472.)

Restated, *g* in *g*(*f*(*n*)) converts the prime decomposition's exponents to coefficients and its multiplication operators to addition: $g(p_1^{a_1} \cdot p_2^{a_2} \ldots \cdot p_k^{a_k}) = a_1 p_1 + a_2 p_2 \ldots + a_k p_k$.

Next, *n* and iteration of *sopf*(*n*) defines the *factorization sequence* function *fs*(*n*). That is, *sopf*(21) = 10, *sopf*(10) = 7, *sopf*(7) = 7. When *sopf*(*n*) = *n*, the procedure terminates, as *n* is thus prime or else equal to 4. While $n \neq 4$ is composite, the process repeats. Hence, *fs*(21) = 21, 10, 7. For another example; *fs*(33122169208733534500868101244643841) = 331221692087335345008681012446438 41, 131463022833, 74147424, 157.

Some Transforms Based on *fs*(n)

We next let n = 1, 2, 3… and use *fs*(n) to transform this sequence in two ways. One substitutes the length (j) of the factorization chain for n and the other substitutes the number (n_j) that terminates it.

Table 1: n = 1, 2, 3…

n_1 =	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
j =	1	1	1	1	2	1	3	3	2	1	2	1	4	4	4	1	4	1	4	3	2
n_j =	2	3	4	5	5	7	5	5	7	11	7	13	5	5	5	17	5	19	5	7	13

Table 1 gives the j sequence as 1, 1, 1, 1, 2, 1, 3, 3, 2, 1, 2, 1, 4, 4, 4, 1, 4, 1, 4, 3, 2… (see Sloane's A002217) and the n_j sequence as 2, 3, 4, 5, 5, 7, 5, 5, 7, 11, 7, 13, 5, 5, 5, 17… (Sloane's A029908). As these OEIS pages show, these particular transforms have been looked at closely. In the patterns below, *fs*(n) applies to other common sequences:

Table 2: n = 1, 2, 3, 5, 8…

n_1 =	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597	2584	4181
j =	1	1	1	3	1	3	2	5	1	5	1	4	5	4	1	4	6
n_j =	2	3	5	5	13	7	19	5	89	5	233	7	7	13	1597	7	5

Table 3: n = 1, 3, 6, 10…

n_1 =	3	6	10	15	21	28	36	45	55	66	78	91	105	120	136	153	171	190
j =	1	2	2	4	3	2	3	2	5	5	5	5	5	5	2	2	4	6
n_j =	3	5	7	5	7	11	7	11	5	5	5	5	5	5	23	23	7	5

Note that for the triangular numbers (above), some vague semblance of order seems to be creeping into the j and n_j patterns. This shows up in the square number patterns (below) as well.

Table 4: n = 1, 4, 9, 16…

n_1 =	4	9	16	25	36	49	64	81	100	121	144	169	196	225	256	289
j =	1	3	3	3	3	5	3	3	5	3	5	6	5	5	5	3
n_j =	4	5	7	7	7	5	7	7	5	13	5	5	5	5	5	19

Of the 100,000+ sequences in the OEIS, are there any for which *fs*(n) generates patterns that are entirely regular and predictable?

Sequence Summation Chains

Our next avenue of exploration adds up all of the elements of an *fs*(*n*) expansion and, in turn, applies *fs*(*n*) to the sum. This creates monotonically increasing chains that terminate only at a prime. We'll refer to these chains as summation chains, and represent them (and the function that produces them) as *s*(*n*). In the examples below, the top line of each table holds the sequence of interest; e.g., *s*(22) = 22, 35, 54, 65, 102, 137.

Table 5: Some Examples of Summation Chains

n_1 =	22	35	54	65	102	137
n_2	13	12	11	18	22	
n_3		7		8	13	
n_4				6		
n_5				5		
n_6						

n_1 =	28	39	74	148	189	224	241
n_2	11	16	39	41	16	17	
n_3		8	16		8		
n_4		6	8		6		
n_5		5	6		5		
n_6			5				

n_1 =	496	570	599
n_2	39	29	
n_3	16		
n_4	8		
n_5	6		
n_6	5		

n_1 =	755	951	2043	2276	3159	3198	3257
n_2	156	320	233	573	28	59	
n_3	20	17		194	11		
n_4	9			99			
n_5	6			17			
n_6	5						

Below are the *s*(*n*) that derive from the first few composite *n*:

***s*(6)** = 6, 11; ***s*(8)** = 8, 19; ***s*(9)** = 9, 20, 40, 51, 91, 131; ***s*(10)** = 10, 17; ***s*(12)** = 12, 19; ***s*(14)** = 14, 34, 53; ***s*(15)** = 15, 34, 53; ***s*(16)** = 16, 35, 54, 65, 102, 137; ***s*(18)** = 18, 37; ***s*(20)** = 20, 40, 51, 91, 131; ***s*(21)** = 21, 38, 76, 99, 116, 183, 266, 305, 406, 482, 759, 796, 1052, 1458, 1498, 1681, 1806, 1896, 2001, 2091, 2152, 2465, 2556, 2656, 2802, 3376, 3694, 5685, 6155, 7538, 11781, 11822, 12173, 12304, 13128, 13871, 14027, 14136, 14195, 14419; ***s*(22)** = 22, 35, 54, 65, 102, 137; ***s*(24)** = 24, 44, 78, 115, 154, 194, 310, 386, 619; ***s*(25)** = 25, 42, 61; ***s*(26)** = 26, 60, 79; ***s*(27)** = 27, 47; ***s*(28)** = 28, 39, 74, 148, 189, 224, 241; ***s*(30)** = 30, 47; ***s*(32)** = 32, 49, 83; ***s*(33)** = 33, 67; ***s*(34)** = 34, 53; ***s*(35)** = 35, 54, 65, 102, 137; ***s*(36)** = 36, 53; ***s*(38)** = 38, 76, 99, 116, 183, 266, 305, 406, 482, 759, 796, 1052, 1458, 1498, 1681, 1806, 1896, 2001, 2091, 2152, 2465, 2556, 2656, 2802, 3376, 3694, 5685, 6155, 7538, 11781, 11822, 12173, 12304, 13128, 13871, 14027, 14136, 14195, 14419; ***s*(39)** = 39, 74, 148, 189, 224, 241; ***s*(40)** = 40, 51, 91, 131; ***s*(42)** = 42, 61; ***s*(44)** = 44, 78, 115, 154, 194, 310, 386, 619; ***s*(45) =** 45, 56, 69, 129, 217, 293.

These numbers may be organized into tables to create new sequences. In table 6 (below), t = the length of a given n's sequence, and t_j = its last term.

Table 6

n =	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
t =	1	1	1	1	2	1	2	6	2	1	2	1	3	3	6	1	2	1	5	40	6
t_j =	2	3	4	5	11	7	19	131	17	11	19	13	53	53	137	17	37	19	131	14419	137

n =	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
t =	1	9	3	3	2	7	1	2	1	3	2	2	5	2	1	39	6	4
t_j =	23	619	61	79	47	241	29	47	31	83	67	53	137	53	37	14419	241	131

It could be interesting to see what happens to the ratio n/t as n grows large. We know for prime values of n that $t = 1$, and hence n/t will approach infinity. But what of composites? For $n = 21$ and $t = 40$, $n/t = 0.525\ldots$, only the second instance in the table where n/t is < 1. Is there, in theory, a non-zero lower bound on this ratio? If a chain somehow manages to thread its way through the infinite minefield of primes that lies beyond it (i.e., fails to terminate), then $\lim_{n\to\infty} {}^{n}\!/_{t} = 0$, but what of finite $s(n)$?

This brings up the question, can it be proved that $s(n)$ always terminates? How can we know that a chain on the order of, say, $s(\text{googolplex}^{\text{googolplex}})$ will inevitably encounter one of the ever scarcer primes?

Problems:

Prove that $s(n)$ terminates for all n.

What are the smallest n/t and n/t_j ratios for $n < 10^7$.

How does the average size of t change as n grows large?